

SYZYGY BUNDLES AND THE WEAK LEFSCHETZ PROPERTY OF ALMOST COMPLETE INTERSECTIONS

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ABSTRACT. Deciding the presence of the weak Lefschetz property often is a challenging problem. Continuing studies in [4, 8, 24], in this work an in-depth study is carried out in the case of Artinian monomial ideals with four generators in three variables. We use a connection to lozenge tilings to describe semistability of the syzygy bundle of such an ideal, to determine its generic splitting type, and to decide the presence of the weak Lefschetz property. We provide results in both characteristic zero and positive characteristic.

1. INTRODUCTION

The *weak Lefschetz property* for a standard graded Artinian algebra A over a field K is a natural property. It says that there is a linear form $\ell \in A$ such that the multiplication map $\times \ell : [A]_i \rightarrow [A]_{i+1}$ has maximal rank for all i (i.e., it is injective or surjective). Its presence implies, for example, restrictions on the Hilbert function and graded Betti numbers of the algebra (see [16, 27]). Recent studies have connected the weak Lefschetz property to many other questions (see, e.g., [1, 5, 13, 23, 26, 30, 31, 35]). Thus, a great variety of tools from representation theory, topology, vector bundle theory, hyperplane arrangements, plane partitions, splines, differential geometry, among others has been used to decide the presence of the weak Lefschetz property (see, e.g., [2, 4, 6, 15, 17, 20, 21, 22, 25, 28, 34]). An important aspect has also been the role of the characteristic of K .

Any Artinian quotient of a polynomial ring in at most two variables has the weak Lefschetz property regardless of the characteristic of K (see [29] and [11, Proposition 2.7]). This is far from true for quotients of rings with three or more variables. Here we consider quotients R/I , where $R = K[x, y, z]$ and I is a monomial ideal containing a power of x, y , and z . If I has only three generators, then R/I has the weak Lefschetz property, provided the base field has characteristic zero (see [34, 33, 36, 12]). We focus on the case, where I has four minimal generators, extending previous work in [4, 8, 24]. To this end we use a combinatorial approach developed in [9, 11] that involves lozenge tilings, perfect matchings, and families of non-intersecting lattice paths. Some of our results have already been used in [26].

In Section 2, we recall the connection between monomial ideals in three variables and so-called triangular regions. We use it to establish sufficient and necessary conditions for a balanced triangular subregion to be tileable (see Corollary 2.4). In Section 3, we show that the tileability of a triangular subregion $T_d(I)$ is related to the semistability of the syzygy bundle of the ideal I (see Theorem 3.3). We further recall the relation between lozenge tilings

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of triangular regions and the weak Lefschetz property. All the results up to this point are true for arbitrary Artinian monomial ideals of R . In Section 4 we consider exclusively Artinian monomial ideals with four minimal generators. Our results on the weak Lefschetz property of R/I are summarized in Theorem 4.10. In particular, they provide further evidence for a conjecture in [24], which concerns the case where R/I is a level algebra. Furthermore, we determine the generic splitting type of the syzygy bundle of I in all cases but one (see Propositions 4.19 and 4.21). In the remaining case we show that determining the generic splitting type is equivalent to deciding whether R/I has the weak Lefschetz property (see Theorem 4.23). This result is independent of the characteristic.

2. TRIANGULAR REGIONS

Besides introducing notation, we recall needed facts from the combinatorial approach to Lefschetz properties developed in [9, 11]. We also establish a new criterion for tileability by lozenges.

Let $R = K[x, y, z]$ be a standard graded polynomial ring over a field K , i.e., $\deg x = \deg y = \deg z = 1$. Unless specified otherwise, K is always an arbitrary field. All R -modules in this paper are assumed to be finitely generated and graded. Let $A = R/I = \bigoplus_{j \geq 0} [A]_j$ be a graded quotient of R . The *Hilbert function* of A is the function $h_A : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $h_A(j) = \dim_K [A]_j$. The *socle* of A , denoted $\text{soc } A$, is the annihilator of $\mathfrak{m} = (x, y, z)$, the homogeneous maximal ideal of R , that is, $\text{soc } A = \{a \in A \mid a \cdot \mathfrak{m} = 0\}$.

2.1. Triangular regions.

Let I be a monomial ideal of R . As R/I is standard graded, the monomials of R of degree $d \in \mathbb{Z}$ that are *not* in I form a K -basis of $[R/I]_d$.

Let $d \geq 1$ be an integer. Consider an equilateral triangle of side length d that is composed of $\binom{d}{2}$ downward-pointing (∇) and $\binom{d+1}{2}$ upward-pointing (\triangle) equilateral unit triangles. We label the downward- and upward-pointing unit triangles by the monomials in $[R]_{d-2}$ and $[R]_{d-1}$, respectively, as follows: place x^{d-1} at the top, y^{d-1} at the bottom-left, and z^{d-1} at the bottom-right, and continue labeling such that, for each pair of an upward- and a downward-pointing triangle that share an edge, the label of the upward-pointing triangle is obtained from the label of the downward-pointing triangle by multiplying with a variable. The resulting labeled triangular region is the *triangular region (of R) in degree d* and is denoted \mathcal{T}_d . See Figure 2.1(i) for an illustration.

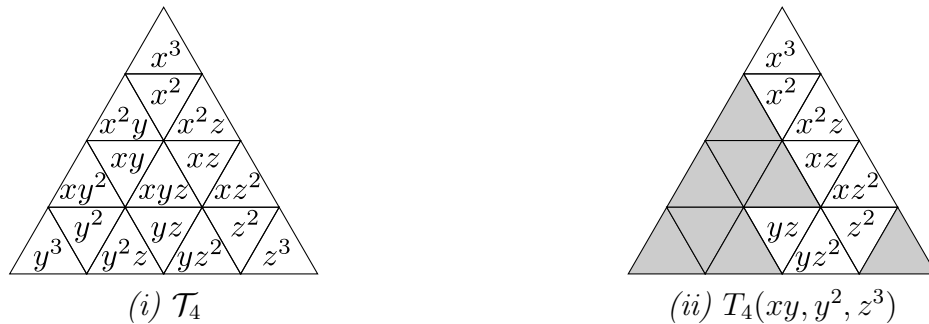


FIGURE 2.1. A triangular region with respect to R and with respect to R/I .

Throughout this manuscript we order the monomials of R with the *graded reverse-lexicographic order*, that is, $x^a y^b z^c > x^p y^q z^r$ if either $a + b + c > p + q + r$ or $a + b + c = p + q + r$ and

the *last* non-zero entry in $(a - p, b - q, c - r)$ is *negative*. For example, in degree 3,

$$x^3 > x^2y > xy^2 > y^3 > x^2z > xyz > y^2z > xz^2 > yz^2 > z^3.$$

Thus in \mathcal{T}_d , see Figure 2.1(iii), the upward-pointing triangles are ordered starting at the top and moving down-left in lines parallel to the upper-left edge.

We generalize this construction to quotients by monomial ideals. Let I be a monomial ideal of R . The *triangular region (of R/I) in degree d* , denoted by $T_d(I)$, is the part of \mathcal{T}_d that is obtained after removing the triangles labeled by monomials in I . Note that the labels of the downward- and upward-pointing triangles in $T_d(I)$ form K -bases of $[R/I]_{d-2}$ and $[R/I]_{d-1}$, respectively. It is more convenient to illustrate such regions with the removed triangles darkly shaded instead of being removed. See Figure 2.1(ii) for an example.

Notice that the regions missing from \mathcal{T}_d in $T_d(I)$ can be viewed as a union of (possibly overlapping) upward-pointing triangles of various side lengths that include the upward- and downward-pointing triangles inside them. Each of these upward-pointing triangles corresponds to a minimal generator of I that has, necessarily, degree at most $d - 1$. We can alternatively construct $T_d(I)$ from \mathcal{T}_d by removing, for each minimal generator $x^a y^b z^c$ of I of degree at most $d - 1$, the *puncture associated to $x^a y^b z^c$* which is an upward-pointing equilateral triangle of side length $d - (a + b + c)$ located a triangles from the bottom, b triangles from the upper-right edge, and c triangles from the upper-left edge. See Figure 2.2 for an example. We call $d - (a + b + c)$ the *side length of the puncture associated to $x^a y^b z^c$* , regardless of possible overlaps with other punctures in $T_d(I)$.

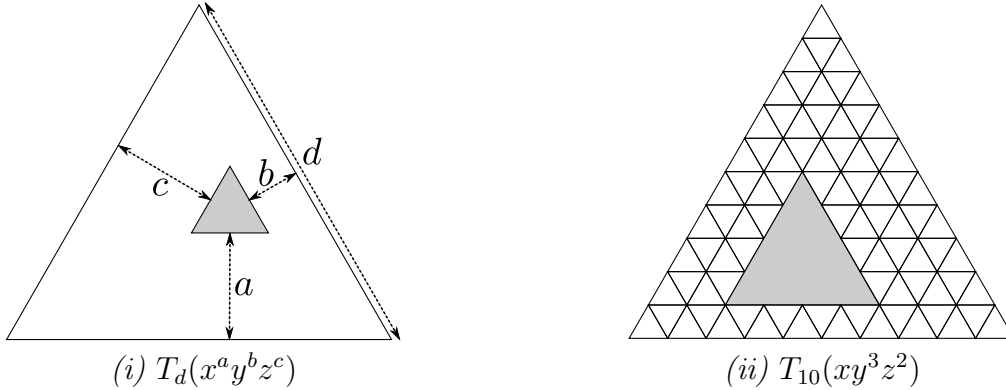


FIGURE 2.2. $T_d(I)$ as constructed by removing punctures.

We say that two punctures *overlap* if they share at least an edge. Two punctures are said to be *touching* if they share precisely a vertex.

2.2. Tilings with lozenges.

A *lozenge* is a union of two unit equilateral triangles glued together along a shared edge, i.e., a rhombus with unit side lengths and angles of 60° and 120° . Lozenges are also called calissons and diamonds in the literature. See Figure 2.3.

Fix a positive integer d and consider the triangular region \mathcal{T}_d as a union of unit triangles. Thus a *subregion* $T \subset \mathcal{T}_d$ is a subset of such triangles. We retain their labels. As above, we say that a subregion T is ∇ -heavy, \triangle -heavy, or *balanced* if there are more downward pointing than upward pointing triangles or less, or if their numbers are the same, respectively. A subregion is *tileable* if either it is empty or there exists a tiling of the region by lozenges such that every triangle is part of exactly one lozenge. Since a lozenge in \mathcal{T}_d is the union of a



FIGURE 2.3. A triangular region $T \subset \mathcal{T}_8$ together with one of its 13 tilings.

downward-pointing and an upward-pointing triangle, and every triangle is part of exactly one lozenge, a tileable subregion is necessarily balanced.

Let $T \subset \mathcal{T}_d$ be any subregion. Given a monomial $x^a y^b z^c$ with degree less than d , the *monomial subregion* of T associated to $x^a y^b z^c$ is the part of T contained in the triangle a units from the bottom edge, b units from the upper-right edge, and c units from the upper-left edge. In other words, this monomial subregion consists of the triangles that are in T and the puncture associated to the monomial $x^a y^b z^c$. See Figure 2.4 for an example.

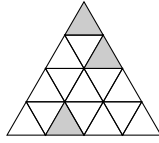


FIGURE 2.4. The monomial subregion of $T_8(x^7, y^7, z^6, xy^4z^2, x^3yz^2, x^4yz)$ (see Figure 2.3) associated to xy^2z .

Replacing a tileable monomial subregion by a puncture of the same size does not alter tileability.

Lemma 2.1. [9, Lemma 2.1] *Let $T \subset \mathcal{T}_d$ be any subregion. If a monomial subregion U of T is tileable, then T is tileable if and only if $T \setminus U$ is tileable.*

Moreover, each tiling of T is obtained by combining a tiling of $T \setminus U$ and a tiling of U .

Let $U \subset \mathcal{T}_d$ be a monomial subregion, and let $T, T' \subset \mathcal{T}_d$ be any subregions such that $T \setminus U = T' \setminus U$. If $T \cap U$ and $T' \cap U$ are both tileable, then T is tileable if and only if T' is, by Lemma 2.1. In other words, replacing a tileable monomial subregion of a triangular region by a tileable monomial subregion of the same size does not affect tileability.

Theorem 2.2. [9, Theorem 2.2] *Let $T = T_d(I)$ be a balanced triangular region, where $I \subset R$ is any monomial ideal. Then T is tileable if and only if T has no ∇ -heavy monomial subregions.*

Let I be a monomial ideal of R whose punctures in \mathcal{T}_d (corresponding to the minimal generators of I having degree less than d) have side lengths that sum to m . Then we define the *over-puncturing coefficient* of I in degree d to be $\mathfrak{o}_d(I) = m - d$. If $\mathfrak{o}_d(I) < 0$, $\mathfrak{o}_d(I) = 0$, or $\mathfrak{o}_d(I) > 0$, then we call I *under-punctured*, *perfectly-punctured*, or *over-punctured* in degree d , respectively.

Let now $T = T_d(I)$ be a triangular region with punctures whose side lengths sum to m . Then we define similarly the *over-puncturing coefficient* of T to be $\mathfrak{o}_d(T) = m - d$. If $\mathfrak{o}_d(T) < 0$, $\mathfrak{o}_d(T) = 0$, or $\mathfrak{o}_d(T) > 0$, then we call T *under-punctured*, *perfectly-punctured*, or *over-punctured*, respectively.

Observe that different monomial ideals can determine the same triangular region of \mathcal{T}_d . Consider, for example, $I_1 = (x^5, y^5, z^5, xyz^2, xy^2z, x^2yz)$ and $I_2 = (x^5, y^5, z^5, xyz)$. Then $T_6(I_1) = T_6(I_2)$, and $\mathfrak{o}_6(I_1) = 3$ but $\mathfrak{o}_6(I_2) = 0$. However, given a triangular region $T = T_d(I)$, there is a unique largest ideal J that is generated by monomials whose degrees are bounded above by $d-1$ and that satisfies $T = T_d(J)$. We call $J(T)$ the *monomial ideal of the triangular region* T . Note that $\mathfrak{o}_d(T) = \mathfrak{o}_d(J(T)) \leq \mathfrak{o}_d(I)$, and equality is true if and only if the ideals I and $J(T)$ are the same in all degrees less than d .

Remark 2.3. If a monomial subregion T of \mathcal{T}_d has no overlapping punctures, then $\mathfrak{o}_d(T)$ is equal to the number of downwards-pointing unit triangles in T minus the number of upward-pointing unit triangles in T .

Perfectly-punctured regions admit a numerical tileability criterion.

Corollary 2.4. *Let $T = T_d(I)$ be a triangular region. Then any two of the following conditions imply the third:*

- (i) T is perfectly-punctured;
- (ii) T has no over-punctured monomial subregions; and
- (iii) T is tileable.

Proof. Suppose T is tileable. Then T has no ∇ -heavy monomial subregions by Theorem 2.2. Thus, every monomial subregion of T is not over-punctured if and only if no punctures of T overlap. Hence (ii) implies (i) by Remark 2.3 because T is balanced. For the converse it is enough to show: If some punctures of T overlap, then T is over-punctured. Indeed, if no punctures overlap, then T is perfectly punctured because T is balanced. So assume two punctures of T overlap. Then the smallest monomial subregion U of T containing these two punctures does not overlap with any other puncture of T and is uniquely tileable. Hence $T \setminus U$ is tileable by Lemma 2.1, and thus $0 \leq \mathfrak{o}_d(T \setminus U) < \mathfrak{o}_d(T)$, as desired.

If T is non-tileable, then T has a ∇ -heavy monomial subregion. Since every ∇ -heavy monomial subregion is also over-punctured, it follows that T has an over-punctured monomial subregion. \square

Any subregion $T \subset \mathcal{T}_d$ can be associated to a bipartite planar graph G that is an induced subgraph of a honeycomb graph (see [9]). We are interested in the bi-adjacency matrix $Z(T)$ of G . This is a zero-one matrix whose determinant enumerates signed lozenge tilings (see [9, Theorem 3.5]). If $T = T_d(I)$ for some monomial ideal I , then $Z(T)$ admits an alternative description. Indeed, consider the multiplication map $\times(x + y + z) : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$. Let $M(d)$ be the matrix to this linear map with respect to the monomial bases of $[R/I]_{d-2}$ and $[R/I]_{d-1}$ in reverse-lexicographic order. Then the transpose of $M(d)$ is the bi-adjacency matrix $Z(T_d(I))$ (see [11, Proposition 4.5]). Here we need only a special case of these results.

Proposition 2.5. *Assume $T = T_d(I) \subset \mathcal{T}_d$ is a non-empty balanced subregion. If $\det Z(T) \in \mathbb{Z}$ is not zero, then T is tileable.*

Proof. Balancedness of T is equivalent to $\dim_K [R/I]_{d-2} = \dim_K [R/I]_{d-1}$. It follows that $Z(T)$ is a square matrix by [11, Proposition 4.5]. Now [9, Theorem 3.5] gives the assertion. \square

We conclude this section with a criterion that guarantees non-vanishing of $\det Z(T)$. To this end we recursively define a puncture of $T \subset \mathcal{T}_d$ to be a *non-floating* puncture if it touches the boundary of \mathcal{T}_d or if it overlaps or touches a non-floating puncture of T . Otherwise we call a puncture a *floating* puncture. For example, the region T in Figure 2.3 has three non-floating

punctures (in the corners) and three floating punctures, two of them are overlapping and have side length two.

Proposition 2.6. [9, Corollary 4.7] *Let T be a tileable triangular region, and suppose all floating punctures of T have an even side length. Then $\text{perm } Z(T) = |\det Z(T)| \neq 0$.*

3. COMBINATORIAL INTERPRETATIONS OF SOME ALGEBRAIC PROPERTIES

In this section, we use the connection to triangular regions to reinterpret some algebraic properties.

3.1. Stability of syzygy bundles.

Throughout this subsection, we assume the characteristic of K is zero.

Let I be an Artinian ideal of $S = K[x_1, \dots, x_n]$ that is minimally generated by forms f_1, \dots, f_m . The *syzygy module* of I is the graded module $\text{syz } I$ that fits into the exact sequence

$$0 \rightarrow \text{syz } I \rightarrow \bigoplus_{i=1}^m S(-\deg f_i) \rightarrow I \rightarrow 0.$$

Its sheafification $\widetilde{\text{syz } I}$ is a vector bundle on \mathbb{P}^{n-1} , called the *syzygy bundle* of I . It has rank $m - 1$.

Semistability is an important property of a vector bundle. Let E be a vector bundle on projective space. The *slope* of E is defined as $\mu(E) := \frac{c_1(E)}{\text{rk}(E)}$. Furthermore, E is said to be *semistable* if the inequality $\mu(F) \leq \mu(E)$ holds for every coherent subsheaf $F \subset E$. If the inequality is always strict, then E is said to be *stable*.

Brenner established a beautiful characterization of the semistability of syzygy bundles to monomial ideals. Since we only consider monomial ideals in this work, the following may be taken as the definition of (semi)stability herein.

Theorem 3.1. [3, Proposition 2.2 & Corollary 6.4] *Let I be an Artinian ideal in $K[x_1, \dots, x_n]$ that is minimally generated by monomials g_1, \dots, g_m , where K is a field of characteristic zero. Then I has a semistable syzygy bundle if and only if, for every proper subset J of $\{1, \dots, m\}$ with at least two elements, the inequality*

$$\frac{d_J - \sum_{j \in J} \deg g_j}{|J| - 1} \leq \frac{-\sum_{i=1}^m \deg g_i}{m - 1}$$

holds, where d_J is the degree of the greatest common divisor of the g_j with $j \in J$. Further, I has a stable syzygy bundle if and only if the above inequality is always strict.

We use Brenner's criterion to rephrase (semi)stability in the case of a monomial ideal of $K[x, y, z]$ in terms of the over-puncturing coefficients of ideals. Note, in particular, that $\mathfrak{o}_d(I) = \sum_{i=1}^m (d - \deg g_i) - d$.

Corollary 3.2. *Let I be an Artinian ideal in $R = K[x, y, z]$ that is minimally generated by monomials g_1, \dots, g_m of degree at most d . For every proper subset J of $\{1, \dots, m\}$ with at least two elements, let I_J be the monomial ideal that is generated by $\{g_j/g_J \mid j \in J\}$, where $g_J = \gcd\{g_j \mid j \in J\}$ has degree d_J .*

Then I has a semistable syzygy bundle if and only if, for every proper subset J of $\{1, \dots, m\}$ with at least two elements, the inequality

$$\frac{\mathfrak{o}_{d-d_J}(I_J)}{|J| - 1} \leq \frac{\mathfrak{o}_d(I)}{m - 1}$$

holds. Furthermore, I has a stable syzygy bundle if and only if the above inequality is always strict.

Proof. Since $\mathfrak{o}_{d-d_J}(I_J) = d(|J| - 1) + d_J - \sum_{j \in J} \deg g_j$, this follows immediately from Theorem 3.1. \square

In order to apply this result we slightly extend the concept of a triangular region $T_d(I)$. Label the vertices in \mathcal{T}_d by monomials of degree d such that the label of each unit triangle is the greatest common divisor of its vertex labels. Then a minimal monomial generator of I with degree d corresponds to a vertex of \mathcal{T}_d that is removed in $T_d(I)$. We consider this removed vertex as a puncture of side length zero. Observe that this is in line with our general definition of the side length of a puncture.

Using Corollary 2.4, we see that semistability is strongly related to tileability of a region.

Theorem 3.3. *Let I be an Artinian ideal in $R = K[x, y, z]$ generated by monomials whose degrees are bounded above by d , and let $T = T_d(I)$. If T is non-empty, then any two of the following conditions imply the third:*

- (i) I is perfectly-punctured;
- (ii) T is tileable; and
- (iii) $\widetilde{\text{syz}}I$ is semistable.

Proof. Assume I is perfectly punctured, that is, $\mathfrak{o}_d(I) = 0$. We will show that T is tileable if and only if $\widetilde{\text{syz}}I$ is semistable.

If T is tileable, then $\mathfrak{o}_d(T) = 0$ which implies $J(T) = I$. Hence no punctures of T overlap. This further implies $I_A = J(T_{I_A})$ for any subset A of the generators of I . Thus $\mathfrak{o}_{d-d_A}(I_A) = \mathfrak{o}_{d-d_A}(T_{I_A}) \leq 0$, since no punctures overlap and every subregion is not over-punctured by Corollary 2.4. Hence, $\widetilde{\text{syz}}I$ is semistable by Corollary 3.2.

If $\widetilde{\text{syz}}I$ is semistable, then $\mathfrak{o}_{d-d_A}(I_A) \leq 0$ holds for any subset A of the generators of I . This implies, in particular, that no punctures of T overlap. Hence $I = J(T)$ and so $\mathfrak{o}_d(T) = \mathfrak{o}_d(I) = 0$. Furthermore, since no pair of punctures overlap, having no over-punctured regions is the same as having no ∇ -heavy regions (see Remark 2.3). Thus, by Corollary 2.4, T is tileable.

Now assume I is not perfectly-punctured, but T is tileable. We have to show that $\widetilde{\text{syz}}I$ is not semistable. Arguing as in the proof of Corollary 2.4, we conclude that T is over-punctured and must have overlapping punctures. Consider two such overlapping punctures of T . Then the smallest monomial subregion U containing these two punctures does not overlap with any other puncture of T with positive side length. Hence $T' = T \setminus U$ is tileable and $0 \leq \mathfrak{o}_{T'} < \mathfrak{o}_d(I)$. If T' is still over-punctured, then we repeat the above replacement procedure until we get a perfectly-punctured monomial subregion of T . Abusing notation slightly, denote this region by T' . Let J be the largest monomial ideal containing I and with generators whose degrees are bounded above by d such that $T' = T_d(J)$. Observe that $\mathfrak{o}_d(J) = \mathfrak{o}_d(T') = 0$.

Notice that a single replacement step above amounts to replacing the triangular region to an ideal I' by the region to the ideal (I', f) , where f is a greatest common divisor of the minimal

generators of I' that correspond to two overlapping punctures. These generators have degrees less than d .

Assume now that T' is empty. Then I has two relatively prime minimal generators, say g_1, g_2 , whose corresponding punctures overlap and are not both contained in a proper monomial subregion of \mathcal{T}_d . Since I is Artinian it has $m \geq 3$ minimal generators. Moreover, all minimal generators of I other than g_1 and g_2 have degree d . It follows that $\mathfrak{o}_d((g_1, g_2)) = \mathfrak{o}_d(I)$. Since $m > 2$, Corollary 3.2 shows that $\widetilde{\text{syz}}I$ is not semistable.

It remains to consider the case where T' is not empty, i.e., J is a proper ideal of R . Let g_1, \dots, g_m and f_1, \dots, f_n be the minimal monomial generators of I and J , respectively. Partition the generating set of I into $F_j = \{g_i \mid g_i \text{ divides } f_j\}$. Notice $f_j = \gcd\{F_j\}$. In particular, $n > 1$ as J is a proper Artinian ideal.

Set $\mathfrak{o}_j = \mathfrak{o}_{d-\deg f_j}((\frac{F_j}{f_j})) = \sum_{g \in F_j} (d - \deg g) - (d - \deg f_j)$. Observe $\mathfrak{o}_j \geq 0$ as the subregion of $T_d(I)$ associated to f_j is tileable, hence not under-punctured. Moreover,

$$\begin{aligned} \mathfrak{o}_d(J) &= \sum_{j=1}^n (d - \deg f_j) - d = \sum_{j=1}^n \left(\sum_{g \in F_j} (d - \deg g) - \mathfrak{o}_j \right) - d \\ &= \sum_{j=1}^n \sum_{g \in F_j} (d - \deg g) - d - \sum_{j=1}^n \mathfrak{o}_j \\ &= \mathfrak{o}_d(I) - \sum_{j=1}^n \mathfrak{o}_j. \end{aligned}$$

As $\mathfrak{o}_d(J) = 0$, we conclude that $\mathfrak{o}_d(I) = \sum_{j=1}^n \mathfrak{o}_j$ and, in particular, $\mathfrak{o}_d(I) \geq \mathfrak{o}_j$ for each j .

Assume $m \cdot \mathfrak{o}_j < \#F_j \cdot \mathfrak{o}_d(I)$ for all j . Then $m \sum_{j=1}^n \mathfrak{o}_j < \mathfrak{o}_d(I) \sum_{j=1}^n \#F_j = \mathfrak{o}_d(I) \cdot m$. But this implies $m \cdot \mathfrak{o}_d(I) < m \cdot \mathfrak{o}_d(I)$, which is absurd. Hence, there is some k such that $m \cdot \mathfrak{o}_k \geq \#F_k \cdot \mathfrak{o}_d(I)$. Since $\mathfrak{o}_d(I) \geq \mathfrak{o}_k$ it follows that $\frac{\mathfrak{o}_k}{\#F_k - 1} > \frac{\mathfrak{o}_d(I)}{m-1}$. Indeed, this is immediate if $\mathfrak{o}_d(I) > \mathfrak{o}_k$. If $\mathfrak{o}_d(I) = \mathfrak{o}_k$, then it is also true because $\#F_k < m$. Now Corollary 3.2 gives that $\widetilde{\text{syz}}I$ is not semistable. \square

We get the following criterion when focusing solely on the triangular region. Recall that $J(T)$ denotes the monomial ideal of a triangular region T as introduced above Remark 2.3.

Corollary 3.4. *Let I be an Artinian ideal in $R = K[x, y, z]$ generated by monomials whose degrees are bounded above by d , and let $T = T_d(I)$. Assume T is non-empty and tileable.*

- (i) *If $I \neq I + J(T)$, then $\widetilde{\text{syz}}I$ is not semistable.*
- (ii) *$\widetilde{\text{syz}}(I + J(T))$ is semistable if and only if T is perfectly-punctured.*

Proof. Note that $I \neq I + J(T)$ implies $\mathfrak{o}_d(I + J(T)) < \mathfrak{o}_d(I)$. Since T is balanced, we get $0 \leq \mathfrak{o}_d(T) = \mathfrak{o}_d(J(T)) = \mathfrak{o}_d(I + J(T))$. Hence Theorem 3.3 gives our assertions. \square

For stability, we obtain the following result.

Proposition 3.5. *Let I be an Artinian ideal in $R = K[x, y, z]$ generated by monomials whose degrees are bounded above by d . If $T = T_d(I)$ is non-empty, tileable, and perfectly-punctured, then $\widetilde{\text{syz}}(I + J(T))$ is stable if and only if every proper monomial subregion of T is under-punctured.*

Proof. We may assume $I = I + J(T)$. As T is perfectly-punctured, we have that $\mathfrak{o}_d(I) = \mathfrak{o}_d(T) = 0$. In particular, no punctures of T overlap. Using Corollary 3.2, we see that $\widetilde{\text{syz}}I$

is stable if and only if $\mathbf{o}_{d-d_J}(T_{d-d_J}(I_J)) < 0$ for all proper subsets J of the set of minimal generators of I . This is equivalent to every proper monomial subregion of T being under-punctured (see Remark 2.3). \square

By the preceding theorem and proposition, we have an understanding of semistability and stability for perfectly-punctured triangular regions. However, when a region is over-punctured and non-tileable more information is needed to infer semistability.

Example 3.6. There are monomial ideals with stable syzygy bundles whose corresponding triangular regions are over-punctured and non-tileable. See Figure 3.1(i) for a specific example.

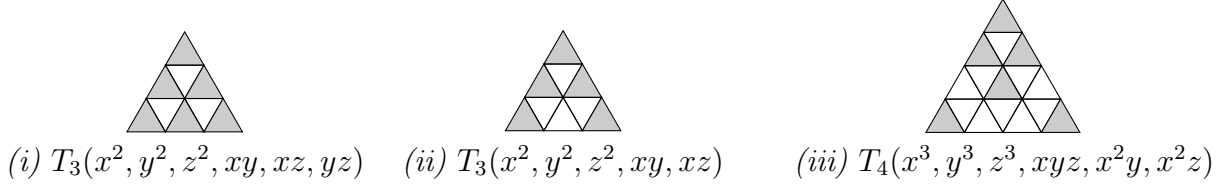


FIGURE 3.1. Over-punctured, non-tileable regions and various levels of stability.

Moreover, the ideal (x^2, y^2, z^2, xy, xz) has a semistable, but non-stable syzygy bundle (the monomial subregion associated to x breaks stability), and the ideal $(x^3, y^3, z^3, xyz, x^2y, x^2z)$ has a non-semistable syzygy bundle (the monomial subregion associated to x^2 breaks semistability). Both of their triangular regions, see Figures 3.1(ii) and (iii), respectively, are over-punctured and non-tileable.

3.2. The weak Lefschetz property.

We recall some results that help decide the presence of the weak Lefschetz property. In fact, one needs only check near a “peak” of the Hilbert function.

Proposition 3.7. [11, Proposition 2.3] *Let $A \neq 0$ be an Artinian standard graded K -algebra, and let ℓ be a general linear form. Suppose A has no non-zero socle elements of degree less than $d - 2$ for some integer $d \geq 0$. Then A has the weak Lefschetz property, provided one of the following conditions is satisfied:*

- (i) $\times \ell : [A]_{d-2} \rightarrow [A]_{d-1}$ is injective and $\times \ell : [A]_{d-1} \rightarrow [A]_d$ is surjective.
- (ii) $\times \ell : [A]_{d-2} \rightarrow [A]_{d-1}$ is bijective.

Moreover, for monomial algebras, it is enough to decide whether the sum of the variables is a Lefschetz element.

Proposition 3.8. [24, Proposition 2.2] *Let $A = R/I$ be a monomial Artinian K -algebra, where K is an infinite field. For any integer d , the following conditions are equivalent:*

- (i) *The multiplication map $\times L : [A]_{d-1} \rightarrow [A]_d$ has maximal rank, where $L \in R$ is a general linear form.*
- (ii) *The multiplication map $\times(x + y + z) : [A]_{d-1} \rightarrow [A]_d$ has maximal rank.*

As pointed out above Proposition 2.5, for a monomial ideal $I \subset K[x, y, z]$, the bi-adjacency matrix $Z(T_d(I))$ can be described using multiplication by $\ell = x + y + z$. We thus get the following criterion for the presence of the weak Lefschetz property, where we consider the entries of $Z(T_d(I))$ as elements of the base field K .

Corollary 3.9. [11, Corollary 4.7] *Let I be an Artinian monomial ideal in $R = K[x, y, z]$. Then R/I has the weak Lefschetz property if and only if, for each positive integer d , the matrix $Z(T_d(I))$ has maximal rank.*

This can be used to infer the weak Lefschetz property in sufficiently large characteristic from its presence in characteristic zero.

Proposition 3.10. [11, Proposition 7.9] *Let R/I be any Artinian monomial algebra such that R/I has the weak Lefschetz property in characteristic zero. If I contains the powers x^a, y^b, z^c , then R/I has the weak Lefschetz property in positive characteristic whenever $\text{char } K > 3^{\frac{1}{2}(\frac{1}{2}(a+b+c)+2)}$.*

4. ARTINIAN MONOMIAL ALMOST COMPLETE INTERSECTIONS

This section presents an in-depth discussion of Artinian monomial ideals of R with exactly four minimal generators. They are called Artinian monomial almost complete intersections. These ideals have been discussed, for example, in [4] and [24, Section 6]. In particular, we will answer some of the questions posed in [24]. Besides addressing the weak Lefschetz property, we discuss the splitting types of the syzygy bundles of these ideals. Particular attention is paid if the characteristic is positive. Some of our results are used in [26] for studying ideals with the Rees property.

Each Artinian ideal of $K[x, y, z]$ with exactly four monomial minimal generators is of the form

$$I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma),$$

where $0 \leq \alpha < a$, $0 \leq \beta < b$, and $0 \leq \gamma < c$, such that at most one of α , β , and γ is zero. If one of α , β , and γ is zero, then $R/I_{a,b,c,\alpha,\beta,\gamma}$ has type two. In this case, the presence of the weak Lefschetz property has already been described in [11]. Thus, throughout this section we assume that the integers α , β , and γ are all positive; this forces $R/I_{a,b,c,\alpha,\beta,\gamma}$ to have Cohen-Macaulay type three. More precisely:

Proposition 4.1. [24, Proposition 6.1] *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ be defined as above. Then R/I has three minimal socle generators. They have degrees $\alpha + b + c - 3$, $a + \beta + c - 3$, and $a + b + \gamma - 3$. In particular, R/I is level if and only if $a - \alpha = b - \beta = c - \gamma$.*

4.1. Presence of the weak Lefschetz property.

Brenner made Theorem 3.1 more explicit in the situation at hand.

Proposition 4.2. [3, Corollary 7.3] *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ be defined as above, and suppose K is a field of characteristic zero. Set $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$. Then I has a semistable syzygy bundle if and only if the following three conditions are satisfied:*

- (i) $\max\{a, b, c, \alpha + \beta + \gamma\} \leq d$;
- (ii) $\min\{\alpha + \beta + c, \alpha + b + \gamma, a + \beta + \gamma\} \geq d$; and
- (iii) $\min\{a + b, a + c, b + c\} \geq d$.

Furthermore, Brenner and Kaid showed that, for almost complete intersections, nonsemistability implies the weak Lefschetz property in characteristic zero.

Proposition 4.3. [4, Corollary 3.3] *Let K be a field of characteristic zero. Then $I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if its syzygy bundle is not semistable.*

The conclusion of this result is not necessarily true in positive characteristic.

Example 4.4. Let $I = I_{5,5,3,1,1,2}$, and thus $d = 6$. Then the syzygy bundle of I is not semistable as $\alpha + \beta + c = 5 < d = 6$. However, the triangular region $T_6(I)$ is balanced and $\det Z(T_6(I)) = 5$. Hence, I does not have the weak Lefschetz property if and only if the characteristic of K is 5.

The following example illustrates that the assumption on the number of minimal generators cannot be dropped in Proposition 4.2.

Example 4.5. Consider the ideal $J = (x^5, y^5, z^5, xy^2z, xyz^2)$ with five minimal generators. Then Theorem 3.2 gives that the syzygy bundle of J is not semistable. Notice that $T_6(J)$ is balanced. However, $\det Z(T_6(J)) = 0$, and so R/J never has the weak Lefschetz property, regardless of the characteristic of K .

The number d in Proposition 4.2 is not assumed to be an integer. In fact, if it is not, then the algebra has the weak Lefschetz property.

Proposition 4.6. [24, Theorem 6.2] *Let K be a field of characteristic zero. Then $I_{a,b,c,\alpha,\beta,\gamma}$ has the weak Lefschetz property if $a + b + c + \alpha + \beta + \gamma \not\equiv 0 \pmod{3}$.*

Again, the conclusion of this result may fail in positive characteristic. Indeed, for the ideal $I_{5,5,3,1,1,2}$ in Example 4.4 we get $d = \frac{17}{3}$, but it does not have the weak Lefschetz property in characteristic 5.

The following result addresses the weak Lefschetz property in the cases that are left out by Propositions 4.3 and 4.6. Its first part extends [24, Lemma 7.1] from level to arbitrary monomial almost complete intersections. Observe that balanced triangular regions correspond to an equality of the Hilbert function in two consecutive degrees, dubbed “twin-peaks” in [24].

Proposition 4.7. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$, and assume $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer. If the syzygy bundle of I is semistable and d is integer, then $T_d(I)$ is perfectly-punctured and balanced.*

Moreover, in this case R/I has the weak Lefschetz property if and only if $\det Z(T_d(I))$ is not zero in K .

Proof. Note that condition (i) in Proposition 4.2 says that $T_d(I)$ has punctures of nonnegative side lengths $d - a, d - b, d - c$, and $d - (\alpha + \beta + \gamma)$. Furthermore, conditions (ii) and (iii) therein are equivalent to the fact that the degree of the least common multiple of any two of the minimal generators of I is at least d , i.e., the punctures of $T_d(I)$ do not overlap. Using the assumption that d is an integer, it follows that $T_d(I)$ is perfectly-punctured, and thus balanced.

Since the punctures of $T_d(I)$ do not overlap, the punctures of $T_{d-1}(I)$ are not overlapping nor touching. Thus we conclude that the degrees of the socle generators of R/I are at least $d - 2$. Hence, Corollary 3.9 and Proposition 3.7 together give that R/I has the weak Lefschetz property if and only if $\det Z(T_d(I))$ is not zero in K . \square

In the situation of Proposition 4.7, the fact that R/I has the weak Lefschetz property implies that $T_d(I)$ is tileable by Proposition 2.5. Tileability remains true even if R/I fails to have the weak Lefschetz property.

Proposition 4.8. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$. If R/I fails to have the weak Lefschetz property in characteristic zero, then $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer and $T_d(I)$ is tileable.*

Proof. By Propositions 4.3 and 4.6, we know that the syzygy bundle of I is semistable and $d = \frac{1}{3}(a+b+c+\alpha+\beta+\gamma)$ is an integer. Hence by Proposition 4.7, $T_d(I)$ is perfectly-punctured. Now we conclude by Theorem 3.3. \square

Before we analyze the presence of the weak Lefschetz property, we need to recall a special type of puncture that has been previously studied by Ciucu, Eisenkölbl, Krattenthaler, and Zare [7].

Remark 4.9. The central puncture is *axes-central* if it is (approximately) equidistant from a corner puncture and the opposite wall, for each of the three punctures. More specifically, suppose $A = d - a$, $B = d - b$, $C = d - c$, and $M = d - (\alpha + \beta + \gamma)$. There are two cases to consider:

- (i) If A , B , and C have the same parity, then the region is of the form

$$T_{A+B+C+M}(x^{B+C+M}, y^{A+C+M}, z^{A+B+M}, x^{\frac{1}{2}(B+C)} y^{\frac{1}{2}(A+C)} z^{\frac{1}{2}(A+B)}).$$

- (ii) If A and B differ in parity from C , then the region is of the form

$$T_{A+B+C+M}(x^{B+C+M}, y^{A+C+M}, z^{A+B+M}, x^{\frac{1}{2}(B+C+1)} y^{\frac{1}{2}(A+C-1)} z^{\frac{1}{2}(A+B)}).$$



(i) The parity of C agrees with A and B . (ii) The parity of C differs from A and B .

FIGURE 4.1. The two prototypical figures with axes-central punctures.

The explicit signed enumerations for these regions can be found in [7, Theorems 1, 2, 4, & 5]. However, the desired consequence for our use is that the signed enumeration is nonzero if and only if not all of A , B , and C are odd. Moreover, if it is nonzero, then the largest prime divisor of the enumeration is bounded above by $d - 1 = A + B + C + M - 1$.

Now, we can decide the presence of the weak Lefschetz property in almost all cases.

Theorem 4.10. *Let $I = I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$ be an Artinian ideal with four minimal generators such that α , β , and γ are all positive. Assume the base field K has characteristic zero, and consider the following conditions:*

- (i) $\max\{a, b, c, \alpha + \beta + \gamma\} \leq d$;
- (ii) $\min\{\alpha + \beta + c, \alpha + b + \gamma, a + \beta + \gamma\} \geq d$;
- (iii) $\min\{a + b, a + c, b + c\} \geq d$; and
- (iv) $d = \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer.

Then the following statements hold:

- (a) *If one of the conditions (i) - (iv) is not satisfied, then R/I has the weak Lefschetz property.*
- (b) *Assume all the conditions (i) - (iv) are satisfied. Then:*

- (1) *The multiplication map $\times(x + y + z) : [R/I]_{j-2} \rightarrow [R/I]_{j-1}$ has maximal rank whenever $j \neq d$.*
- (2) *The algebra R/I has the weak Lefschetz property if one of the following conditions is satisfied:*
 - (I) *Condition (ii) is an equality.*
 - (II) *$a + b + c + \alpha + \beta + \gamma$ is divisible by 6.*
 - (III) *$c = \frac{1}{2}(a + b + \alpha + \beta + \gamma)$.*
 - (IV) *The region $T_d(I)$ has an axes-central puncture (see Remark 4.9) and one of $d - a, d - b, d - c$, and $d - (\alpha + \beta + \gamma)$ is not odd.*
 - (V) *$a = b$, $\alpha = \beta$, and c or γ is even.*
- (3) *The algebra R/I fails to have the weak Lefschetz property if one of the following conditions is satisfied:*
 - (IV') *The region $T_d(I)$ has an axes-central puncture (see Remark 4.9) and all of $d - a, d - b, d - c$, and $d - (\alpha + \beta + \gamma)$ are odd; or*
 - (V') *$a = b$, $\alpha = \beta$, and both c and γ are odd.*

Proof. Assertion (a) follows from Propositions 4.2, 4.3, and 4.6.

Consider now the claims in part (b). Then Proposition 4.7 gives that R/I has the weak Lefschetz property if and only if $\det Z(T_d(I))$ is not zero.

The assumptions in (b) guarantee that the punctures of $T = T_d(I)$ do not overlap and the degrees of the socle generators of R/I are at least $d - 2$. Then condition (I) implies that the puncture to the generator $x^\alpha y^\beta z^\gamma$ touches another puncture, whereas condition (II) says that this puncture has an even side length. In either case, R/I has the weak Lefschetz property by Proposition 2.6.

The proof of (b)(1) uses the Grauert-Mülich splitting theorem. We complete this part below Proposition 4.21.

The remaining assertions all follow from results in [10] and [11], when combined with Proposition 4.7:

(III). The condition $c = \frac{1}{2}(a + b + \alpha + \beta + \gamma)$ is equivalent to $d - c = 0$. After taking into account all lozenges forced by the puncture to $x^\alpha y^\beta z^\gamma$, the remaining subregion of $T_d(I)$ is a hexagon, and so $\det Z(T_d(I)) \neq 0$ (see, e.g., Proposition 2.6).

(IV) and (IV'). Use [7, Theorems 1, 2, 4, & 5], as mentioned in Remark 4.9.

(V) and (V'). Use the results in [10]. □

Notice that Theorem 4.10(b)(1) says that, for almost monomial complete intersections, the multiplication map can fail to have maximal rank in at most one degree.

Remark 4.11. (i) Theorem 4.10 can be extended to fields of sufficiently positive characteristic by using Proposition 3.10. This lower bound on the characteristic can be improved whenever one knows the determinant of $Z(T_d(I))$.

(ii) Question 8.2(2c) in [24] asked if there exist non-level almost complete intersections which never have the weak Lefschetz property. The almost complete intersection $I = I_{3,5,5,1,2,2} = (x^3, y^5, z^5, xy^2z^2)$ is not level and never has the weak Lefschetz property, regardless of field characteristic, as $\det Z(T_6(I)) = 0$.

4.2. Level almost complete intersections.

In the previous subsection, we considered one way of centralizing the inner puncture of a triangular region associated to a monomial almost complete intersection. We called such

punctures “axes-central.” In this section, we consider another method of centralizing the inner puncture of such a triangular region. It turns out this method of centralization is equivalent to the algebra being level.

Consider the ideal $I = I_{a,b,c,\alpha,\beta,\gamma}$ as above. Let d be an integer and assume that $T = T_d(I)$ has one floating puncture. We say the inner puncture of T is a *gravity-central puncture* if the vertices of the puncture are each the same distance from the puncture opposite to it (see Figure 4.2).

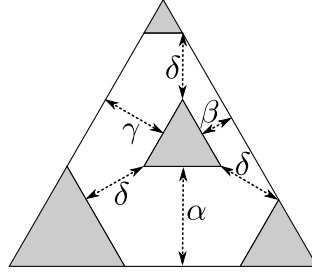


FIGURE 4.2. A prototypical figure with a gravity-central puncture.

Lemma 4.12. *Let $I = I_{a,b,c,\alpha,\beta,\gamma}$. Then $T_d(I)$ has a gravity-central puncture if and only if R/I is a level algebra.*

Proof. The defining property for the distances is $(d-b) + (d-c) - \alpha = (d-a) + (d-c) - \beta = (d-a) + (d-b) - \gamma$. This is equivalent to the condition in Proposition 4.1 that R/I is level, i.e., $a - \alpha = b - \beta = c - \gamma$. \square

Level almost complete intersections were studied extensively in [24, Sections 6 and 7]. In particular, Migliore, Miró-Roig, and the second author proposed a conjectured characterization for the presence of the weak Lefschetz property for such algebras. We recall it here, though we present it in a different, but equivalent, form to better elucidate the reasoning behind it.

Conjecture 4.13. [24, Conjecture 6.8] *Let $I = I_{\alpha+t,\beta+t,\gamma+t,\alpha,\beta,\gamma}$ be an ideal of $R = K[x, y, z]$, where K has characteristic zero, $0 < \alpha \leq \beta \leq \gamma \leq 2(\alpha + \beta)$, $t \geq \frac{1}{3}(\alpha + \beta + \gamma)$, and $\alpha + \beta + \gamma$ is divisible by three. If $(\alpha, \beta, \gamma, t)$ is not $(2, 9, 13, 9)$ or $(3, 7, 14, 9)$, then R/I fails to have the weak Lefschetz property if and only if t is even, $\alpha + \beta + \gamma$ is odd, and $\alpha = \beta$ or $\beta = \gamma$. Furthermore, R/I fails to have the weak Lefschetz property in the two exceptional cases.*

The necessity part of this conjecture was proven in [24, Corollary 7.4]) by showing that R/I does not have the weak Lefschetz property if t is even, $\alpha + \beta + \gamma$ is odd, and $\alpha = \beta$ or $\beta = \gamma$. This result is covered by Theorem 4.10(b)(3)(V') because the region is mirror symmetric. It remained open to establish the presence of the weak Lefschetz property. Theorem 4.10 does this in many new cases.

Proposition 4.14. *Consider the ideal $I = I_{\alpha+t,\beta+t,\gamma+t,\alpha,\beta,\gamma}$ as given in Conjecture 4.13. Then R/I has the weak Lefschetz property if one of the following conditions is satisfied:*

- (i) t and $\alpha + \beta + \gamma$ have the same parity; or
- (ii) t is odd and $\alpha = \beta = \gamma$ is even.

Proof. We apply Theorem 4.10 with $d = t + \frac{2}{3}(\alpha + \beta + \gamma)$. Then the side length of the inner puncture of $T_d(I)$ is $t - \frac{1}{3}(\alpha + \beta + \gamma)$. Hence (i) follows from Theorem 4.10(b)(II). Claim (ii) is a consequence of Theorem 4.10(b)(IV) as the given condition implies the inner puncture is axes-central. \square

Remark 4.15. Conjecture 4.13 remains open in two cases, both of which are conjectured to have the weak Lefschetz property:

- (i) t even, $\alpha + \beta + \gamma$ is odd, and $\alpha < \beta < \gamma$; and
- (ii) t odd, $\alpha + \beta + \gamma$ is even, and $\alpha \leq \beta$ or $\beta \leq \gamma$.

Notice that $T = T_d(I_{a,b,c,\alpha,\beta,\gamma})$ is simultaneously axis- and gravity-central precisely if either $a = b = c$ and $\alpha = \beta = \gamma$, or $a = b + 2 = c + 1$ and $\alpha = \beta + 2 = \gamma + 1$. In the former case, the weak Lefschetz property in characteristic zero is completely characterized below, strengthening [24, Corollary 7.6].

Corollary 4.16. *Let $I = I_{a,a,a,\alpha,\alpha,\alpha} = (x^a, y^a, z^a, x^\alpha, y^\alpha, z^\alpha)$, where $a > \alpha$. Then R/I fails to have the weak Lefschetz property in characteristic zero if and only if α and a are odd and $a \geq 2\alpha + 1$.*

Proof. If $a < 2\alpha$, then R/I has the weak Lefschetz property by Theorem 4.10(a).

Assume now $a \geq 2\alpha$. Then R/I fails the weak Lefschetz property if α and a are odd by [24, Corollary 7.6] (or Theorem 4.10(b)(3)(V')). Otherwise, R/I has this property by Proposition 4.14. \square

For $a \geq 2\alpha$, the triangular region $T_{a+\alpha}(I)$ was considered by Krattenthaler in [19]. He described a bijection between cyclically symmetric lozenge tilings of the region and descending plane partitions with specific conditions.

4.3. Splitting type and regularity.

The generic splitting type of a vector bundle on projective space is an important invariant. However, its computation is often challenging. In this section we consider the splitting type of the syzygy bundles of monomial almost complete intersections in R . These are rank three bundles on the projective plane. For the remainder of this section we assume K is an infinite field.

Let $I = I_{a,b,c,\alpha,\beta,\gamma}$ as above. Recall from Section 3.1 that the syzygy module $\text{syz } I$ of I is defined by the exact sequence

$$0 \longrightarrow \text{syz } I \longrightarrow R(-\alpha - \beta - \gamma) \oplus R(-a) \oplus R(-b) \oplus R(-c) \longrightarrow I \longrightarrow 0,$$

and the syzygy bundle $\widetilde{\text{syz}} I$ on \mathbb{P}^2 of I is the sheafification of $\text{syz } I$. Its restriction to any line H of \mathbb{P}^2 splits as $\mathcal{O}_H(p) \oplus \mathcal{O}_H(q) \oplus \mathcal{O}_H(r)$. The triple (p, q, r) depends on the choice of the line H , but is the same for all general lines. This latter triple is called the *generic splitting type* of $\widetilde{\text{syz}} I$. Since I is a monomial ideal, Proposition 3.8 implies that the generic splitting type (p, q, r) can be determined if we restrict to the line defined by $\ell = x + y + z$.

For computing the generic splitting type of $\widetilde{\text{syz}} I$, we use the observation that $R/(I, \ell) \cong S/J$, where $S = K[x, y]$, and $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$. Define an S -module $\text{syz } J$ by the exact sequence

$$(4.1) \quad 0 \longrightarrow \text{syz } J \longrightarrow S(-\alpha - \beta - \gamma) \oplus S(-a) \oplus S(-b) \oplus S(-c) \longrightarrow J \longrightarrow 0$$

using the, possibly non-minimal, set of generators $\{x^a, y^b, (x+y)^c, x^\alpha y^\beta (x+y)^\gamma\}$ of J . Then $\text{syz } J \cong S(p) \oplus S(q) \oplus S(r)$, where (p, q, r) is the generic splitting type of the vector bundle $\widetilde{\text{syz}} I$. The Castelnuovo-Mumford regularity of the ideal J is $\text{reg } J = 1 + \text{reg } S/J$.

For later use we record the following facts.

Remark 4.17. Adopt the above notation. Then the following statements hold:

- (i) Using, for example, the Sequence (4.1), one gets $-(p+q+r) = a+b+c+\alpha+\beta+\gamma$.
- (ii) If any of the generators of J is extraneous, then the degree of that generator is one of $-p$, $-q$, or $-r$.
- (iii) As the regularity of J is determined by the Betti numbers of S/J , we obtain that $\text{reg } J + 1 = \max\{-p, -q, -r\}$ if the Sequence (4.1) is a minimal free resolution of J .

Before moving on, we prove a technical but useful lemma.

Lemma 4.18. *Let $S = K[x, y]$, where K is a field of characteristic zero. Consider the ideal $\mathfrak{a} = (x^a, y^b, x^\alpha y^\beta (x+y)^\gamma)$ of S , and assume that the given generating set is minimal. Then $\text{reg } \mathfrak{a}$ is*

$$-1 + \max \left\{ a + \beta, b + \alpha, \min \left\{ a + b, a + \beta + \gamma, b + \alpha + \gamma, \left\lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\rceil \right\} \right\}.$$

Proof. We proceed in three steps.

First, considering the minimal free resolution of the ideal $(x^a, y^b, x^\alpha y^\beta)$, we conclude

$$\text{reg}(x^a, y^b, x^\alpha y^\beta) = -1 + \max\{a + \beta, b + \alpha\}.$$

Second, the algebra $S/(x^a, y^b)$ has the strong Lefschetz property in characteristic zero (see, e.g., [16, Proposition 4.4]). Thus, the Hilbert function of $S/(x^a, y^b, (x+y)^\gamma)$ is

$$\dim_K [S/(x^a, y^b, (x+y)^\gamma)]_j = \max\{0, \dim_K [S/(x^a, y^b)]_j - \dim_K [S/(x^a, y^b)]_{j-\gamma}\}.$$

By analyzing when the difference becomes non-positive, we get that

$$(4.2) \quad \text{reg}(x^a, y^b, (x+y)^\gamma) = -1 + \min \left\{ a + b, a + \gamma, b + \gamma, \left\lceil \frac{1}{2}(a + b + \gamma) \right\rceil \right\}.$$

Third, notice that

$$(x^a, y^b, x^\alpha y^\beta (x+y)^\gamma) : x^\alpha y^\beta = (x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma).$$

Hence, multiplication by $x^\alpha y^\beta$ induces the short exact sequence

$$0 \rightarrow [S/(x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma)](-\alpha - \beta) \xrightarrow{\times x^\alpha y^\beta} S/\mathfrak{a} \rightarrow S/(x^a, y^b, x^\alpha y^\beta) \rightarrow 0.$$

It implies

$$\text{reg } \mathfrak{a} = \max\{\alpha + \beta + \text{reg}(x^{a-\alpha}, y^{b-\beta}, (x+y)^\gamma), \text{reg}(x^a, y^b, x^\alpha y^\beta)\}.$$

Using the first two steps, the claim follows. \square

Recall that Proposition 4.2 gives a characterization of the semistability of the syzygy bundle $\widetilde{\text{syz}} I_{a,b,c,\alpha,\beta,\gamma}$, using only the parameters a, b, c, α, β , and γ . We determine the splitting type of $\widetilde{\text{syz}} I_{a,b,c,\alpha,\beta,\gamma}$ for the nonsemistable and the semistable cases separately.

4.3.1. *Nonsemistable syzygy bundle.*

We first consider the case when the syzygy bundle is not semistable, and therein we distinguish four cases. It turns out that in three cases, at least one of the generators of the ideal J is extraneous.

Proposition 4.19. *Consider the ideal $I = I_{a,b,c,\alpha,\beta,\gamma} = (x^a, y^b, z^c, x^\alpha y^\beta z^\gamma)$ with four minimal generators. Assume that the base field K has characteristic zero and, without loss of generality, that $a \leq b \leq c$. Set $d := \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$, and denote by (p, q, r) the generic splitting type of $\widetilde{\text{syz}}I$. Assume that $\widetilde{\text{syz}}I$ is not semistable. Then:*

(i) *If $\min\{\alpha + \beta + \gamma, c\} \geq a + b - 1$, then*

$$(p, q, r) = (-c, -\alpha - \beta - \gamma, -a - b).$$

(ii) *Assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$ and*

$$\frac{1}{2}(a + b + c) \leq \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma, \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\}.$$

Then

$$(p, q, r) = (-\alpha - \beta - \gamma, -\left\lceil \frac{1}{2}(a + b + c) \right\rceil, -\left\lfloor \frac{1}{2}(a + b + c) \right\rfloor).$$

(iii) *Assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$ and*

$$\frac{1}{2}(a + b + \alpha + \beta + \gamma) \leq \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma, \frac{1}{2}(a + b + c) \right\}.$$

Then

$$(p, q, r) = (-c, q, -a - b - \alpha - \beta - \gamma + q),$$

$$\text{where } -q = \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, \left\lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\rceil \right\}.$$

(iv) *Assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$ and*

$$-s = \min \{a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma\} < \min \left\{ \frac{1}{2}(a + b + \alpha + \beta + \gamma), \frac{1}{2}(a + b + c) \right\}.$$

Then

$$(p, q, r) = \left(\left\lfloor \frac{1}{2}(-3d - s) \right\rfloor, \left\lceil \frac{1}{2}(-3d - s) \right\rceil, s \right).$$

Proof. Set

$$\mu = \min \left\{ a + b, a + \beta + \gamma, b + \alpha + \gamma, c + \beta + \gamma, \frac{1}{2}(a + b + \alpha + \beta + \gamma), \frac{1}{2}(a + b + c) \right\}.$$

Using $a \leq b \leq c$, [3, Theorem 6.3] implies that the maximal slope of a subsheaf of $\widetilde{\text{syz}}I$ is $-\mu$. Since $\widetilde{\text{syz}}I$ is not semistable, we have $\mu < d$ (see Proposition 4.2). Moreover, the generic splitting type of $\widetilde{\text{syz}}I$ is determined by the minimal free resolution of $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$ as a module over $S = K[x, y]$. We combine both approaches to determine the generic splitting type.

Since $\text{reg}(x^a, y^b) = a + b - 1$, all polynomials in S whose degree is at least $a + b - 1$ are contained in (x^a, y^b) . Hence, $J = (x^a, y^b)$ if $\min\{\alpha + \beta + \gamma, c\} \geq a + b - 1$, and the claim in case (i) follows by Remark 4.17.

For the remainder of the proof, assume $\min\{\alpha + \beta + \gamma, c\} \leq a + b - 2$. Then $a + b > \frac{1}{2}(a + b + c)$, and thus $\mu \neq a + b$.

In case (ii), it follows that $\mu = \frac{1}{2}(a + b + c)$ and $c \leq \alpha + \beta + \gamma$, and thus $c \leq a + b - 2$. Using Equation (4.2), we conclude that

$$\operatorname{reg}(x^a, y^b, (x + y)^c) = -1 + \min \left\{ a + b, \left\lceil \frac{1}{2}(a + b + c) \right\rceil \right\} = -1 + \left\lceil \frac{1}{2}(a + b + c) \right\rceil.$$

Observe now that $d > \mu = \frac{1}{2}(a + b + c)$ is equivalent to $\alpha + \beta + \gamma > \frac{1}{2}(a + b + c)$. This implies $\alpha + \beta + \gamma > \operatorname{reg}(x^a, y^b, (x + y)^c)$, and thus $J = (x^a, y^b, (x + y)^c)$. Using Remark 4.17 again, we get the generic splitting type of $\widetilde{\operatorname{syz}}I$ as claimed in (ii).

Consider now case (iii). Then $d > \mu = \frac{1}{2}(a + b + \alpha + \beta + \gamma)$, which gives $c > \frac{1}{2}(a + b + \alpha + \beta + \gamma)$. The second assumption in this case also implies $\frac{1}{2}(a + b + \alpha + \beta + \gamma) \leq a + \beta + \gamma$, which is equivalent to $b + \alpha \leq a + \beta + \gamma$ and also to $b + \alpha \leq \frac{1}{2}(a + b + \alpha + \beta + \gamma)$. Similarly, we have that $\frac{1}{2}(a + b + \alpha + \beta + \gamma) \leq b + \alpha + \gamma$, which is equivalent to $a + \beta \leq b + \alpha + \gamma$ and also to $a + \beta \leq \frac{1}{2}(a + b + \alpha + \beta + \gamma)$. It follows that

$$\max\{a + \beta, b + \alpha\} \leq \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\}.$$

Hence Lemma 4.18 yields

$$\begin{aligned} \operatorname{reg}(x^a, y^b, x^\alpha y^\beta (x + y)^\gamma) = \\ -1 + \min \left\{ a + \beta + \gamma, b + \alpha + \gamma, \left\lceil \frac{1}{2}(a + b + \alpha + \beta + \gamma) \right\rceil \right\} < c. \end{aligned}$$

This shows that $(x + y)^c \in (x^a, y^b, x^\alpha y^\beta (x + y)^\gamma) = J$. Setting $-q = 1 + \operatorname{reg} J$, Remark 4.17 provides the generic splitting type in case (iii).

Finally consider case (iv). Then $\mu = -s$, and μ is equal to the degree of the least common multiple of two of the minimal generators of I . In fact, $-\mu = s$ is the slope of the syzygy bundle $\mathcal{O}_{\mathbb{P}^2}(s)$ of the ideal generated by these two generators. Thus, the Harder-Narasimhan filtration (see [18, Definition 1.3.2]) gives an exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(s) \rightarrow \widetilde{\operatorname{syz}}I \rightarrow \mathcal{E} \rightarrow 0,$$

where \mathcal{E} is a semistable torsion-free sheaf on \mathbb{P}^2 of rank two and first Chern class $-a - b - c - \alpha - \beta - \gamma - s = -3d - s$. Its bidual \mathcal{E}^{**} is a stable vector bundle. Thus, by the theorem of Grauert and Müllich (see [14] or [32, Corollary 1 of Theorem 2.1.4]), its generic splitting type is $(\lfloor \frac{1}{2}(-3d - s) \rfloor, \lceil \frac{1}{2}(-3d - s) \rceil)$. Now the claim follows by restricting the above sequence to a general line of \mathbb{P}^2 . \square

We have seen that the ideal $J = (x^a, y^b, (x + y)^c, x^\alpha y^\beta (x + y)^\gamma)$ has at most three minimal generators in the cases (i) - (iii) of the above proposition. In the fourth case, the associated ideal $J \subset S$ may be minimally generated by four polynomials.

Example 4.20. Consider the ideal

$$I = I_{4,5,5,3,1,1} = (x^4, y^5, z^5, x^3 y z).$$

Then the corresponding ideal J is minimally generated by $x^4, y^5, (x + y)^5$, and $x^3 y (x + y)$. The syzygy bundle of $\widetilde{\operatorname{syz}}I$ is not semistable, and its generic splitting type is $(-7, -6, -6)$ by Proposition 4.19(iv).

4.3.2. Semistable syzygy bundle.

Order the entries of the generic splitting type (p, q, r) of the semistable syzygy bundle $\widetilde{\text{syz}}I$ such that $p \leq q \leq r$. In this case, the splitting type determines the presence of the weak Lefschetz property if the characteristic of K is zero (see [4, Theorem 2.2]). The following result is slightly more precise.

Proposition 4.21. *Let K be a field of characteristic zero, and assume the ideal $I = I_{a,b,c,\alpha,\beta,\gamma}$ has a semistable syzygy bundle. Set $k = \lfloor \frac{1}{3}(a+b+c+\alpha+\beta+\gamma) \rfloor$. Then the generic splitting type of $\widetilde{\text{syz}}I$ is*

$$(p, q, r) = \begin{cases} (-k-1, -k, -k) & \text{if } a+b+c+\alpha+\beta+\gamma = 3k+1; \\ (-k-1, -k-1, -k) & \text{if } a+b+c+\alpha+\beta+\gamma = 3k+2; \\ (-k, -k, -k) & \text{if } a+b+c+\alpha+\beta+\gamma = 3k \text{ and} \\ & R/I \text{ has the weak Lefschetz property;} \\ (-k-1, -k, -k+1) & \text{if } a+b+c+\alpha+\beta+\gamma = 3k \text{ and} \\ & R/I \text{ fails to have the weak Lefschetz property.} \end{cases}$$

Proof. The Grauert-Mülich theorem [14] gives that $r - q$ and $q - p$ are both nonnegative and at most 1. Moreover, p, q , and r satisfy $a+b+c+\alpha+\beta+\gamma = -(p+q+r)$ (see Remark 4.17(i)). This gives the result if $k \neq d = \frac{1}{3}(a+b+c+\alpha+\beta+\gamma)$.

It remains to consider the case when $k = d$. Then $(-k, -k, -k)$ and $(-k-1, -k, -k+1)$ are the only possible generic splitting types. By Proposition 4.2(i), the minimal generators of the ideal $J = (x^a, y^b, (x+y)^c, x^\alpha y^\beta (x+y)^\gamma)$ have degrees that are less than d . Hence $\text{reg } J = d$ if and only if the splitting type of $\widetilde{\text{syz}}I$ is $(-d-1, -d, -d+1)$. Since $\dim_K [R/I]_{d-2} = \dim_K [R/I]_{d-1}$, using Proposition 4.7, we conclude that $\text{reg } J \geq d$ if and only if R/I does not have the weak Lefschetz property. \square

We are ready to add the missing piece in the proof of Theorem 4.10.

Completion of the proof of Theorem 4.10(b)(1).

We have just seen that the ideal $J = (x^a, y^b, (x+y)^c, x^\alpha y^\beta (x+y)^\gamma)$ has regularity d if R/I fails the weak Lefschetz property. This implies that the multiplication map $\times(x+y+x) : [R/I]_{j-2} \rightarrow [R/I]_{j-1}$ is surjective whenever $j > d$. Moreover, since the minimal generators of J have degrees that are less than d , we have the exact sequence

$$0 \longrightarrow S(-d+1) \oplus S(-d) \oplus S(-d-1) \longrightarrow S(-\alpha-\beta-\gamma) \oplus S(-a) \oplus S(-b) \oplus S(-c) \longrightarrow J \longrightarrow 0.$$

In the above proof of Theorem 4.10 we saw that the four punctures of $T_d(I)$ do not overlap and that $T_d(I)$ is balanced. Hence $T_{d-1}(I)$ has 3 more downward-pointing than upward-pointing triangles, that is,

$$\dim_K [R/I]_{d-2} = \dim_K [R/I]_{d-3} + 3.$$

It follows that the multiplication map in the exact sequence

$$[R/I]_{d-3} \longrightarrow [R/I]_{d-2} \longrightarrow S/J \longrightarrow 0$$

is injective because $\dim_K [S/J]_{d-2} = 3$. Hence $\times(x+y+x) : [R/I]_{j-2} \rightarrow [R/I]_{j-1}$ is injective whenever $j \leq d-1$. \square

The second author would like to thank the authors of [26]; it was during a conversation in the preparation of that paper that he learned about the use of the Grauert-Mülich theorem

for an alternative way of deducing the injectivity of the map $[R/I]_{d-3} \rightarrow [R/I]_{d-2}$ in the above argument if the characteristic of K is zero.

Example 4.22. Consider the ideal $I_{7,7,7,3,3,3} = (x^7, y^7, z^7, x^3y^3z^3)$. It never has the weak Lefschetz property, by Theorem 4.10(vii). The bundle $\widetilde{\text{syz}}I_{7,7,7,3,3,3}$ has generic splitting type $(-11, -10, -9)$. Notice that the similar ideal $I_{6,7,8,3,3,3} = (x^6, y^7, z^8, x^3y^3z^3)$ has the weak Lefschetz property in characteristic zero as $\det N_{6,7,8,3,3,3} = -1764$. The generic splitting type of $\widetilde{\text{syz}}I_{6,7,8,3,3,3}$ is $(-10, -10, -10)$.

We summarize part of our results for the case where I is associated to a tileable triangular region. In particular, if K is an infinite field of arbitrary characteristic, then the splitting type can be used to determine the presence of the weak Lefschetz property.

Theorem 4.23. *Let $I = I_{a,b,c,\alpha,\beta,\gamma} \subset R = K[x, y, z]$, where K is an infinite field of arbitrary characteristic. Assume I satisfies conditions (i)–(iv) in Theorem 4.10 and $d := \frac{1}{3}(a + b + c + \alpha + \beta + \gamma)$ is an integer. Then the following conditions are equivalent:*

- (i) *The algebra R/I has the weak Lefschetz property.*
- (ii) *The determinant of $Z(T_d(I))$ (i.e., the enumeration of signed perfect matchings of the bipartite graph $G(T_d(I))$) is not zero in K .*
- (iii) *The generic splitting type of $\widetilde{\text{syz}}I$ is $(-d, -d, -d)$.*

Proof. Regardless of the characteristic of K , the arguments for Proposition 4.7 show that $T_d(I)$ is balanced. Moreover, the degrees of the socle generators of R/I are at least $d - 2$ as shown in Theorem 4.10(b)(1). Hence, Proposition 3.7 gives that R/I has the weak Lefschetz property if and only if the multiplication map

$$\times(x + y + z) : [R/I]_{d-2} \rightarrow [R/I]_{d-1}$$

is bijective. Now, Corollary 3.9 yields the equivalence of Conditions (i) and (ii).

As above, let (p, q, r) be the generic splitting type of $\widetilde{\text{syz}}I$, where $p \leq q \leq r$, and let $J \subset S$ be the ideal such that $R/(I, x + y + z) \cong S/J$. The above multiplication map is bijective if and only if $\text{reg } J = d - 1$. Since $\text{reg } J + 1 = -r$ and $p + q + r = -3d$, it follows that $\text{reg } J = d - 1$ if and only if $(p, q, r) = (-d, -d, -d)$. Hence, conditions (i) and (iii) are equivalent. \square

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